Math 352 HW. # 6

Homework problems are taken from "Real Analysis" by N. L. Carothers. The problems are color coded to indicate level of difficulty. The color green indicates an elementary problem, which you should be able to solve effortlessly. Yellow means that the problem is somewhat harder. Red indicates that the problem is hard. You should attempt the hard problems especially.

Unless the contrary is explicitly stated, all numbers that are mentioned in these exercises are understood to be real.

1. Let $f,\ g \in R_\alpha[a,\ b]$ be two functions that satisfy $f \le g$, show that $\int_a^b f d\alpha \le \int_a^b g d\alpha$.

2. If $f,\ g \in R_\alpha[a,\ b]$, show that $f + g \in R_\alpha[a,\ b]$ and that $\int_a^b (f+g) d\alpha = \int_a^b f d\alpha + \int_a^b g d\alpha$.

3. If $f \in R_\alpha[a,\ b]$, show that $|f| \in R_\alpha[a,\ b]$ and that $\left|\int_a^b f d\alpha\right| \le \int_a^b |f| d\alpha$. [Hint: $U(|f|,\ P) - L(|f|,\ P) \le U(f,\ P) - L(f,\ P)$. Why?]

4. If $f,\ g \in R_\alpha[a,\ b]$, is $fg \in R_\alpha[a,\ b]$? How about f^2?

5. If $f \in R_\alpha[a,\ b]$, show that $f \in R_\alpha[c,\ d]$ for every subinterval $[c, d]$ of $[a, b]$. Moreover, $\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha$ for every $a < c < b$. In fact, if any two of these integrals exist, then so does the third and the equation above still holds.

6. Given $f \in R_\alpha[a,\ b]$, define $F(x) = \int_a^x f d\alpha$ for $a \le x \le b$. Show that $F \in BV[a,\ b]$. If α is continuous, show that $F \in C[a,\ b]$.

7. If $\int_a^b f d\alpha = 0$ for every $f \in C[a,\ b]$, show that α is constant.

8. If $f \in R_\alpha[a,\ b]$ and $U(f,\ P) - L(f,\ P) \le \varepsilon$ for some partition P, show that $\left|\sum_{i=1}^n f(t_i)\Delta\alpha_i - \int_a^b f d\alpha\right| < \varepsilon$, where t_i is any point in $[x_{i-1},\ x_i]$.

9. Suppose there exists a number I with the property that, given any $\varepsilon > 0$, there is a partition P such that $\left|\sum_{i=1}^{n} f(t_i)\Delta\alpha_i - I\right| < \varepsilon$, where t_i is any point in $[x_{i-1},\ x_i]$. Show that $f \in R_\alpha[a,\ b]$ and $I = \int_a^b f d\alpha$.

10. If $U(f,\ P) - L(f,\ P) < \varepsilon$, show that $\sum_{i=1}^{n} |f(t_i) - f(s_i)|\Delta\alpha_i < \varepsilon$ for any choice of points $s_i,\ t_i \in [x_{i-1},\ x_i]$.

11. If f and α share a common-sided discontinuity in [a, b], show that f is not in $R_\alpha[a,\ b]$.

12. Show that $\cap\{R_\alpha[a,\ b]:\ \alpha$ increasing$\}$ = C[a, b].

13. If α is continuous, show that $\int_a^b f d\alpha$ does not depend on the values of f at any *finite* number of points. Is this still true if we change "finite" to "countable"? Explain.

14. Construct a nonconstant increasing function α and a nonzero continuous function $f \in R_\alpha[a,\ b]$ such that $\int_a^b |f(x)| d\alpha = 0$. Is it possible to choose α to also be continuous? Explain.

15. If f is continuous on [a, b], and if $f(x_0) \neq 0$ for some x_0, show that $\int_a^b |f(x)| dx \neq 0$. Conclude that $\|f\| = \int_a^b |f(x)| dx$ defines a norm on C[a, b]. Does it define a norm on all of $R[a,\ b]$? Explain.

16. Give an example of a sequence of Riemann integrable functions on [0, 1] that converges pointwise to a nonintegrable function.